# 4.4 Modeling and Optimization

## Strategy for Solving Max-Min Problems

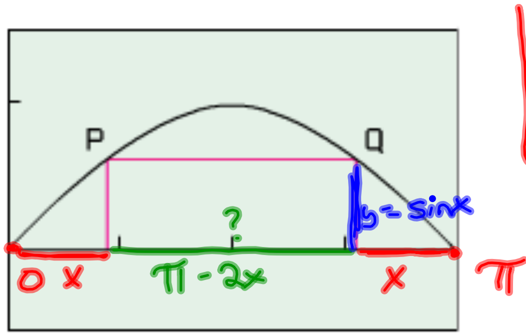
1. **Understand the Problem** Read the problem carefully. Identify the information you need to solve the problem.
2. **Develop a Mathematical Model of the Problem** Draw pictures and label the parts that are important to the problem. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, write a function whose extreme value gives the information sought.
3. **Graph the Function** Find the domain of the function. Determine what values of the variable make sense in the problem.
4. **Identify the Critical Points and Endpoints** Find where the derivative is zero or fails to exist.
5. **Solve the Mathematical Model** If unsure of the result, support or confirm your solution with another method.
6. **Interpret the Solution** Translate your mathematical result into the problem setting and decide whether the result makes sense.

**EXAMPLE 1 Using the Strategy**

Find two numbers whose sum is 20 and whose product is as large as possible.

**EXAMPLE 2 Inscribing Rectangles**

A rectangle is to be inscribed under one arch of the sine curve (Figure 4.36). What is the largest area the rectangle can have, and what dimensions give that area?




[0, π] by [−0.5, 1.5]

**Figure 4.36** A rectangle inscribed under one arch of $y = \sin x$. (Example 2)

$A = l \cdot w$

$A = (\pi - 2x)(\sin x)$

domain: $0 \le x \le \frac{\pi}{2}$

$A' = (\pi - 2x)\cos x - 2\sin x$

use grapher to find where $A' = 0$

$x \approx .71$ max

$A = (\pi - 2(.71))(\sin .71) \approx 1.12 \text{ units}^2$

It is not possible to solve the equation $A'(x) = 0$ using algebraic methods. We can use the graph of $A$ (Figure 4.37a) to find the maximum value and where it occurs. Or, we can use the graph of $A'$ (Figure 4.37b) to find where the derivative is zero, and then evaluate $A$ at this value of $x$ to find the maximum value. The two $x$-values appear to be the same, as they should.

$l = (\pi - 2(.71)) \approx 1.72 \text{ units}$

$w = \sin(.71) \approx .65 \text{ units}$

## Examples from Business and Industry

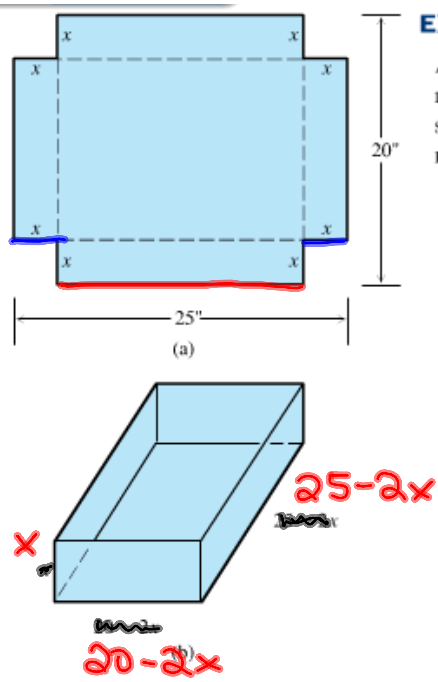



**Figure 4.38** An open box made by cutting the corners from a piece of tin. (Example 3)

**EXAMPLE 3 Fabricating a Box**

An open-top box is to be made by cutting congruent squares of side length $x$ from the corners of a 20- by 25-inch sheet of tin and bending up the sides (Figure 4.38). How large should the squares be to make the box hold as much as possible? What is the resulting maximum volume?

domain: $0 < x < 10$

$V = l \cdot w \cdot h$

$V = (25-2x)(20-2x)(x)$

$V = 4x^3 - 90x^2 + 500x$

$V' = 12x^2 - 180x + 500$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{180 \pm \sqrt{(180)^2 - 4(12)(500)}}{2(12)} \approx 3.68 \; \& \; \cancel{11.32}$$

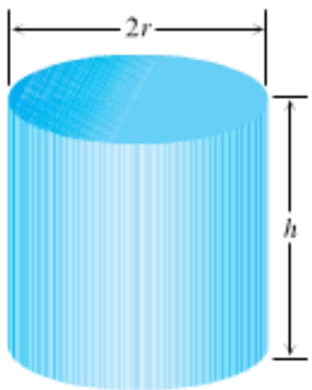
Squares should be 3.68 x 3.68

$\max V \approx 4(3.68)^3 - 90(3.68)^2 + 500(3.68)$

$\approx 820.53 \text{ in}^3$

**EXAMPLE 4 Designing a Can**

You have been asked to design a one-liter oil can shaped like a right circular cylinder (see Figure 4.40 on the next page). What dimensions will use the least material?




1 liter = 1000 cm$^3$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

$$h = \frac{1000}{\pi r^2}$$

$$= \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2}$$

$$= \frac{2(500)^{3/3}}{\pi^{3/3}} \cdot \frac{\pi^{2/3}}{500^{2/3}}$$

$$= 2 \cdot \frac{500^{1/3}}{\pi^{1/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

## Examples from Economics

$r(x)$ = the revenue from selling $x$ items,

$c(x)$ = the cost of producing the $x$ items,

$p(x) = r(x) - c(x)$ = the profit from selling $x$ items.

$\frac{dr}{dx}$ = marginal revenue, $\frac{dc}{dx}$ = marginal cost, $\frac{dp}{dx}$ = marginal profit.

$$p'(x) = r'(x) - c'(x)$$

**EXAMPLE 5 Maximizing Profit**

Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where $x$ represents thousands of units. Is there a production level that maximizes profit? If so, what is it?

$$p(x) = 9x - (x^3 - 6x^2 + 15x)$$
$$= 9x - x^3 + 6x^2 - 15x$$
$$\longrightarrow = -x^3 + 6x^2 - 6x$$

$$r'(x) = c'(x)$$

$$9 = 3x^2 - 12x + 15$$
$$0 = 3x^2 - 12x + 6$$
$$= \frac{12 \pm \sqrt{(12)^2 - 4(3)(6)}}{2(3)}$$
$$\approx \boxed{3.414} \text{ and } 0.586$$

**THEOREM 7 Minimizing Average Cost**

The production level (if any) at which average cost is smallest is a level at which the average cost equals the marginal cost.

**EXAMPLE 6 Minimizing Average Cost**

Suppose $c(x) = x^3 - 6x^2 + 15x$, where $x$ represents thousands of units. Is there a production level that minimizes average cost? If so, what is it?

avg cost : $\frac{c(x)}{x} = x^2 - 6x + 15$

marginal cost: $c'(x) = 3x^2 - 12x + 15$

$x^2 - 6x + 15 = 3x^2 - 12x + 15$

$2x^2 - 6x = 0$ $\quad x = 0$

$2x(x-3) = 0$ $\quad$ or $\boxed{x = 3}$